

2019 BC #5
(no calculator)

(a)

$$f(x) = \frac{1}{x^2 - 2x + k} = (x^2 - 2x + k)^{-1}$$

$$f'(x) = -\frac{1}{(x^2 - 2x + k)^2} (2x - 2)$$

$$f'(0) = -\frac{-2}{k^2} = \frac{2}{k^2} = 6 \Rightarrow k = \sqrt{\frac{1}{3}} \quad \text{Note: only the positive square root since } k > 0$$

(b)

$$\int_0^1 \frac{1}{x^2 - 2x - 8} dx = \int_0^1 \frac{1}{(x-4)(x+2)} dx$$

$$\text{Partial fractions: } \frac{1}{(x-4)(x+2)} = \frac{A}{x-4} + \frac{B}{x+2}$$

$$1 = A(x+2) + B(x-4)$$

$$x = -2 : B = -\frac{1}{6}$$

$$x = 4 : A = \frac{1}{6}$$

$$\begin{aligned} \int_0^1 \frac{1}{x^2 - 2x - 8} dx &= \int_0^1 \frac{1}{(x-4)(x+2)} dx = \frac{1}{6} \int_0^1 \left(\frac{1}{x-4} - \frac{1}{x+2} \right) dx \\ &= \frac{1}{6} \left(\ln|x-4| - \ln|x+2| \right) \Big|_0^1 \\ &= \frac{1}{6} \ln \left| \frac{x-4}{x+2} \right| \Big|_0^1 \\ &= \frac{1}{6} (\ln 1 - \ln 2) = \boxed{-\frac{1}{6} \ln 2} \end{aligned}$$

(c)

$$\int_0^2 \frac{1}{x^2 - 2x + 1} dx = \int_0^2 \frac{1}{(x-1)^2} dx \quad \text{Improper integral}$$

$$= \lim_{a \rightarrow 1^-} \int_0^a (x-1)^{-2} dx + \lim_{b \rightarrow 1^+} \int_b^2 (x-1)^{-2} dx$$

$$= \lim_{a \rightarrow 1^-} -\left(x-1 \right)^{-1} \Big|_0^a + \lim_{b \rightarrow 1^+} -\left(x-1 \right)^{-1} \Big|_b^2$$

$$= -\lim_{a \rightarrow 1^-} \left[\frac{1}{a-1} - \frac{1}{0-1} \right] - \lim_{b \rightarrow 1^+} \left[\frac{1}{2-1} - \frac{1}{b-1} \right]$$

$$= \lim_{a \rightarrow 1^-} \frac{1}{a-1} \text{ diverges so } \boxed{\text{the integral diverges}}.$$